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ON HOMOTOPIES OF TRANSIENT VECTOR FIELDS

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§1. INTRODUCTION

IN THIS PAPER we prove an analogue for vector fields of a theorem of Levine [1] concerning the possibility of eliminating certain singularities of maps. At the end of the paper we will suggest that this result might be used as the first step towards a proof of the classical Poincaré conjecture which states that a closed smooth 3-manifold which is homotopy equivalent to the 3-sphere is actually diffeomorphic to the 3-sphere (see [2]).

The following notation is used throughout the paper: M is a compact, connected C^∞ $(n+1)$-manifold with non-empty boundary V, N is the double of M and $\mathfrak{X}(M)$ is the set of C^∞ vector fields on M with the C^∞ topology. D^{n+1} is the closed $(n+1)$-disk in $\mathbf{R}^{n+1}$ with center at 0 and radius 1 and S^n is its boundary, the n-sphere.

We recall for use in this paper some of the results in [3]. $X \in \mathfrak{X}(M)$ is called *transient* if each integral curve for X leaves M in finite positive and negative time. The set of transient vector fields on M is always non-empty and open in $\mathfrak{X}(M)$. Let $\beta: M \to \mathbf{R}$ be any smooth function such that $\beta^{-1}(0) = V$ and 0 is a regular value of β. The particular choice of β does not matter in what follows. $X \in \mathfrak{X}(M)$ is called *b-generic* if for every non-negative integer k the map

$$(\beta, X\beta, \ldots, X^k\beta): M \to \mathbf{R}^{k+1}$$

has 0 as a regular value. (Here vector fields are thought of as derivations.) The set of b-generic vector fields is open and dense in $\mathfrak{X}(M)$. Furthermore, there are local normal forms for b-generic vector fields. If X is g-generic and $m_o \in V$ is such that

$$X\beta(m_o) = \ldots = X^p\beta(m_o) = 0$$

but $X^{p+1}\beta(m_o) \neq 0$, then given any extension, $\tilde{X}$, of X to N we can choose coordinate functions

$$x_1, \ldots, x_n, t: U \to \mathbf{R},$$

where U is a neighborhood of m_o in N, so that

(a) $$x_1(m_o) = \ldots = x_n(m_o) = t(m_o) = 0$$

(b) $$\tilde{X}|U = \frac{\partial}{\partial t}$$

(c) $M \cap U$ is the set of points where one of the functions

$$\pm\left[t^{p+1} + \sum_{j=1}^{p} x_j t^{j-1}\right]$$

is non-negative.

Note that the integer p must be such that $0 \le p \le n$. (In [3] b-generic vector fields were called vector fields which satisfy condition L. They exhibit "generic" behavior on the boundary of M, hence the term b-generic.)

We give special names to certain kinds of points in V relative to a given b-generic vector field, X, on M. The names come from the behavior with respect to V and M of the integral curve for X through m_o. The behavior can be seen by looking at the normal forms.

(1) $m_o \in V$ is called a *transverse point* for X if $X\beta(m_o) \neq 0$. The transverse points are divided into two types: *transverse entrance* and *transverse exit* points.

(2) $m_o \in V$ is called a *tangency point* for X if $X\beta(m_o) = 0$ but $X^2\beta(m_o) \neq 0$. There are two types of tangency point: *interior tangencies* and *exterior tangencies.*

(3) $m_o \in V$, is called an *inflection point* for X if $X\beta(m_o) = X^2\beta(m_o) = 0$ but $X^3\beta(m_o) \neq 0$. Again there are two types: *entering inflections* and *exiting inflections.*

Now suppose $\dim M = 3$. Given a b-generic vector field on M, the set of non-transverse points in V is a finite union of disjointly embedded circles. Each of these circles is a component of the boundary of one region of transverse entrance points and one region of transverse exit points. Furthermore, on every circle there is a finite number of inflection points each of which separates an arc of interior tangencies from an arc of exterior tangencies. In particular, the number of inflection points on each of the circles must be even.

THEOREM A. *If* $\dim M = 3$, *then every transient vector field on M is homotopic through transient vector fields to a b-generic transient vector field with no inflection points.*

If we did not ask that something about the behavior of the vector field inside of M—such as being transient—remain unchanged, then this theorem would be trivial, but could not be used in the way to be suggested in §3.

§2. ELIMINATING INFLECTIONS WHEN $\dim M = 3$

This section is entirely devoted to the proof of Theorem A. By the density of the set of b-generic vector fields and the openness of the set of transient vector fields, we can assume that the vector field we start with in Theorem A is a b-generic transient vector field. The theorem will be proved by the following.

LEMMA. *Suppose that* $\dim M = 3$ *and X is a b-generic transient vector field on M. If A is a closed arc of non-transverse points for X such that the endpoints are the only inflection points on A and U is an open neighborhood of A which contains no inflection points except the endpoints of A, then there is a homotopy of X, constant off U, through transient vector fields to a b-generic transient vector field with no inflection points at all in U.*

Proof. Let $\tilde{X}$ be an extension of X to N and let $\tilde{U}$ be an open neighborhood of A in N such that $\tilde{U} \cap M = U$. We will specify homotopies of X which are constant off U by giving deformations with compact support of $V \cap \tilde{U}$ on $\tilde{U}$ while leaving $\tilde{X}$ fixed. We can do this because such deformations of $V \cap \tilde{U}$ induce deformations of M in N and $\tilde{X}$ restricted to a deformation of M in N is equivalent to a vector field on M which is homotopic to X. It will be easy to check, simply by keeping track of entrance and exit points of integral curves, that all the vector fields in the homotopies of X are transient.

We will use pictures heavily in the proof.

The basic picture is that of an inflection point. Using the normal form for inflections, there are coordinates x, y, t on a neighborhood in N of any inflection point such that the inflection point is at the origin, $\tilde{X} = \partial/\partial t$ locally and M is given locally as the set of points where one of the two functions

$$\pm(t^3 + yt + x)$$

is non-negative. Thus Fig. 1 represents the local situation. The only thing actually shown in this picture is a piece of the boundary of M. The trajectories of the vector field, which are vertical lines directed upward, and the side of the boundary on which M lies are left to the imagination of the reader. The essence of this picture can be given conveniently by showing the projection of the set of non-transverse points onto the x-y plane and a few representative vertical sections. Thus Fig. 2 is equivalent to Fig. 1. This method of representation will be used in more complicated situations with more than one inflection point present.

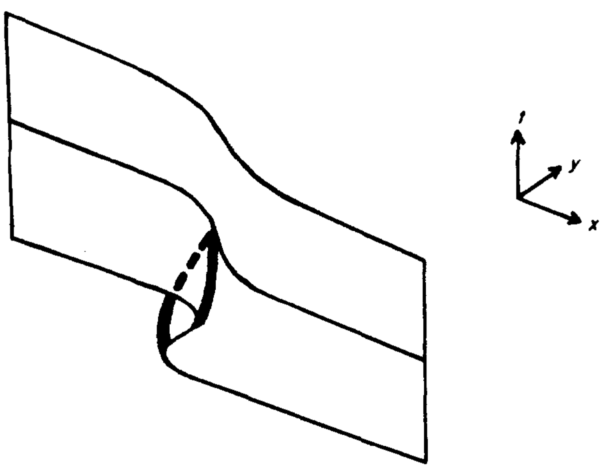

Fig. 1.

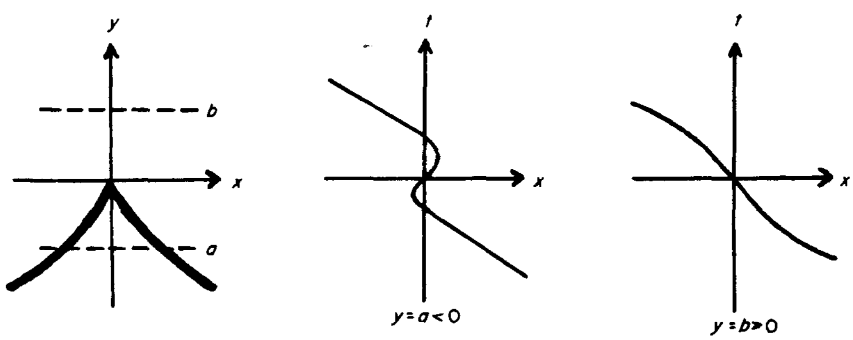

Fig. 2.

Getting back to the proof of the lemma, there are two possibilities which must be treated differently. The first is that both inflection points on A are of the same type, i.e. entering or exiting. In this case the inflection points can be run together so that they both disappear. The second case, in which the inflection points are of opposite type, is taken care of by reducing it to two copies of the first case through the device of a preliminary deformation which introduces two new inflection points of opposite type on the interior of A without creating any other inflection points.

Case I

The endpoints of A are inflection points of the same type. In this case there are still four different possibilities: the inflection points may be entering or exiting and the interior of A may consist of interior or exterior tangency points. These various possibilities are not essentially different, however, and all of them can be handled at the same time.

We claim that there are coordinates x, y, t on a neighborhood $\tilde{W}$ of A in N such that $\tilde{W} \subset \tilde{U}$, $\tilde{X}|\tilde{W}$ is parallel to $\partial/\partial t$ and Fig. 3 represents $V \cap \tilde{W}$ sitting in $\tilde{W}$. We get all the possibilities mentioned above by letting $M \cap \tilde{W}$ lie on either side of $V \cap \tilde{W}$ and by letting $\tilde{X}|\tilde{W}$ be $+\partial/\partial t$ or $-\partial/\partial t$.

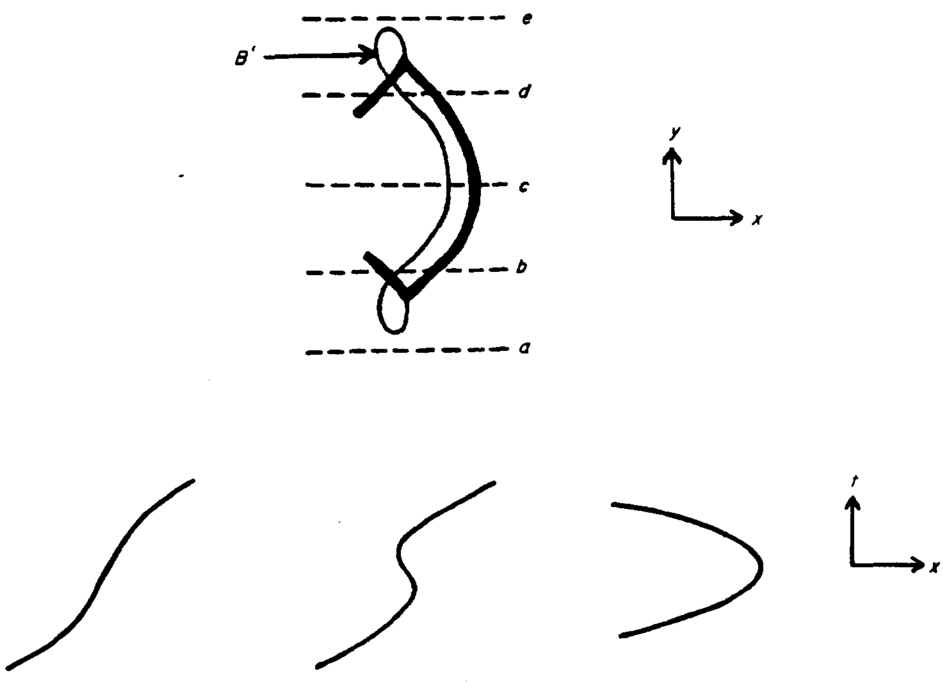

Fig. 3.

In the top part of the figure the heavy curve is the projection onto the x-y plane of the set of non-transverse points in $\tilde{W}$ and the light arc, B', which joins the two inflection points is to be used in a moment as a track on which to run the inflections together. Because of the normal forms for tangency points and inflection points, the only aspect of this representation which we really need to worry about justifying is that the sections $y = b$ and $y = d$ are the same. But the sections could differ only by being reflections of each other across a horizontal line. The fact that both inflections are of the same type disposes of this problem.

Now we show how to make the inflections eliminate each other. The closed arc B' which joins the two inflections in the x-y plane has a unique lift to a smooth arc, B, on $V \cap \tilde{W}$ which again joins the two inflections. B lies in the lowest part of $V \cap \tilde{W}$ whenever there is a choice and does not touch the set of non-transverse points except at its endpoints. Figure 4 gives a representation of what happens in a neighborhood, smaller than $\tilde{W}$, of the arc B. The coordinates used in Fig. 4 are of course different from those used in Fig. 3. Finally, the configuration of V near B which is represented in Fig. 4 can easily be deformed so that the inflections are run together and there are just two arcs of tangency points of opposite type left. Figure 5 represents the end

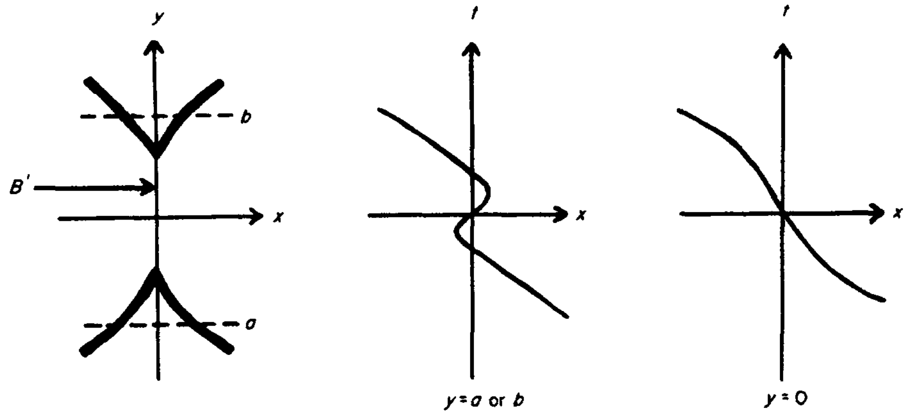

Fig. 4.

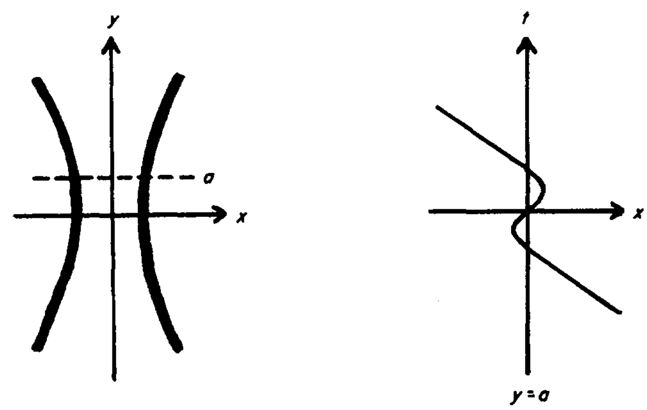

Fig. 5.

result of this deformation. The homotopy of real valued functions

$$\beta_\sigma(x, y, t) = t^3 + (\sigma - y^2)t + x \qquad -1 \leq \sigma \leq 1$$

gives a smooth model for this deformation. $\beta_\sigma^{-1}(0)$ has the configuration of Fig. 4 when $\sigma > 0$ and the configuration of Fig. 5 when $\sigma < 0$. There is clearly no problem about carrying out this deformation inside of $\tilde{U}$.

Case II

The endpoints of A are inflection points of opposite type. The homotopy of real valued functions

$$\gamma_\sigma(x, y, t) = \sigma[t^4 - t^2 + \epsilon yt] + (1 - \sigma)t^2 + x,$$

where $0 \leq \sigma \leq 1$ and $\epsilon = \pm 1$, gives a smooth model for a deformation which introduces a pair of inflection points with opposite type on an arc of tangency points. The vector field stays fixed as $\partial/\partial t$ and for each value of σ, the set $\gamma_\sigma^{-1}(0)$ gives a configuration of V. Essentially, we are dealing with Thom's swallow's tail. When $\sigma = 0$, we just have the normal form for a tangency point, which is interior or exterior according to which side of V the interior of M lies on. Figure 6 represents $\gamma_\sigma^{-1}(0)$ when $\sigma = 1$ and $\epsilon = +1$. Letting $\epsilon = -1$ has the effect of switching the sections $y = b$ and $y = c$ which means exchanging the positions of the entering and exiting

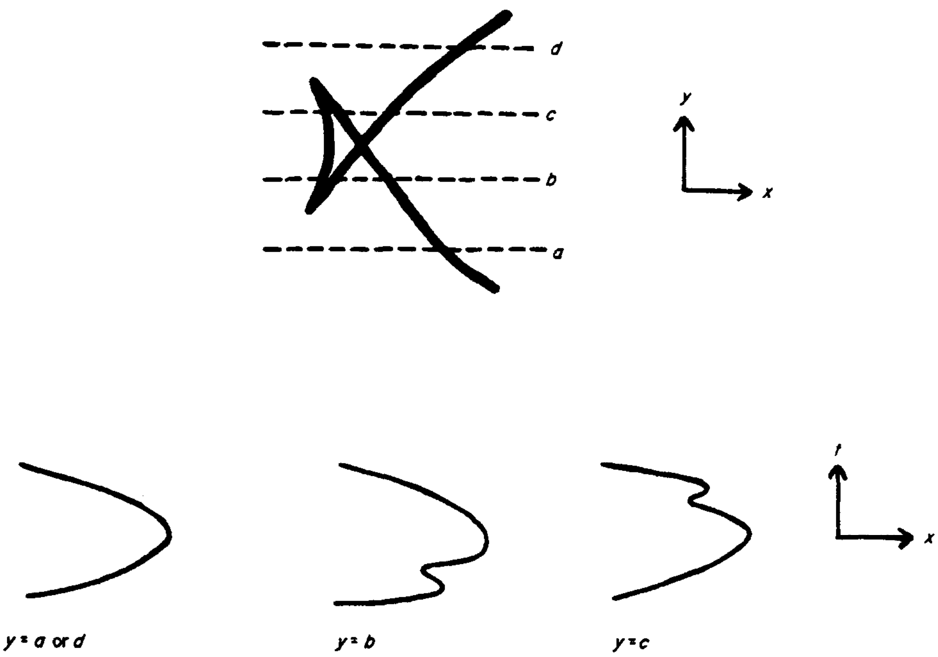

Fig. 6.

inflections. The deformation can be localized to take place in any neighborhood of a tangency point. This discussion shows that when the endpoints of A are inflection points of opposite type we can introduce two more inflections of opposite type on the interior of A in such a way that inflection points of the same type have no other inflections points between them. This leaves us with two copies of Case I so we are finished with the proof of the lemma and of Theorem A.

§3. ON THE 3-DISK CONJECTURE

PROPOSITION 3.1. *For $n \geq 0$, every transient vector field X on D^{n+1} is homotopic through transient vector fields to the b-generic transient vector field $(\partial/\partial z)/D^{n+1}$, where z is any non-zero linear functional on* $\mathbf{R}^{n+1}$.

Proof. X can easily be modified (by a homotopy of transient vector fields which remains fixed outside of a neighborhood of 0) so that $X = (\partial/\partial z)$ on a neighborhood of 0. Thus we assume that $X = (\partial/\partial z)$ on D_ϵ^{n+1}, the closed $(n+1)$-disk with center at 0 and radius ϵ, for some ϵ such that $0 < \epsilon \leq 1$. Now define a homotopy X_τ by

$$X_\tau(y) = X((1 + \tau(\epsilon - 1))y), \qquad y \in D^{n+1}, \qquad 0 \leq \tau \leq 1.$$

Clearly $X_0 = X$ and $X_1 = (\partial/\partial z)|D^{n+1}$. Also, X_τ is transient for each τ because X_τ is equivalent to the restriction of X to the closed disk with center at 0 and radius equal to $1 + \tau(\epsilon - 1)$.||

Note that the set of non-transverse points for $(\partial/\partial z)|D^{n+1}$ is an equator of S^n and consists entirely of exterior tangency points. As a sort of converse to Proposition 3.1 we have the following

PROPOSITION 3.2. *If V is diffeomorphic to S^n and M admits a b-generic transient vector field X such that the set of non-transverse points for X in V is an embedded $(n-1)$-sphere consisting entirely of exterior tangency points, then M is homeomorphic to D^{n+1}.*

Proof. Let E be the set of transverse entrance points for X. Then by Brown's Generalized Schoenfliess Theorem [4], $\bar{E}$, the closure of E, is homeomorphic to D^n, so E is homeomorphic to $\mathbb{R}^n$. Next we will show that int M, the interior of M, is diffeomorphic to $E \times \mathbb{R}$, and hence is homeomorphic to $\mathbb{R}^{n+1}$. This will finish the proof because M can be shrunk into its interior so, since V is S^n, we can apply the Generalized Schoenfliess Theorem again to conclude that M is homeomorphic to D^{n+1}.

Let E' be the set obtained by moving each point $e \in E$ along the trajectory through e for a time equal to one half the time it takes for that trajectory to move through M. Then E' is diffeomorphic to E and is a submanifold of int M. Furthermore, E' is a cross section to $X|\text{int } M$, i.e. E' is transverse to $X|\text{int } M$ and meets each trajectory of $X|\text{int } M$ exactly once. E' is transverse to $X|\text{int } M$ because E is transverse to X. E' meets each trajectory of $X|\text{int } M$ once and only once because the trajectories of X itself either reduce to single points of exterior tangency in V or are closed arcs which meet V only at their endpoints, one of which is in E. Thus int M is diffeomorphic to $E' \times \mathbb{R}$ which is diffeomorphic to $E \times \mathbb{R}$. ||

Together these two propositions indicate that transient vector fields might be used in the study of manifolds. This idea was more or less suggested by Thom in [5]. Since every transient vector field is the gradient of a function with no critical points [3], this might be thought of as being dual to the classical use of Morse functions in the study of manifolds. In this program one must deal with tangencies of a vector field to the boundary of a manifold instead of critical points of a vector field in the interior of a manifold.

The "3-disk conjecture", which is equivalent to the classical Poincaré conjecture, says that every compact, contractible, C^∞ 3-manifold, W, with boundary S^2 is diffeomorphic to D^3 (see [2]). Since the set of transient vector fields on W is non-empty, it seems reasonable to try to prove this conjecture by attempting to show that every transient vector field on W is homotopic through transient vector fields to a transient vector field which satisfies the hypothesis of Proposition 3.2. Theorem A is clearly a possible step in this direction.

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